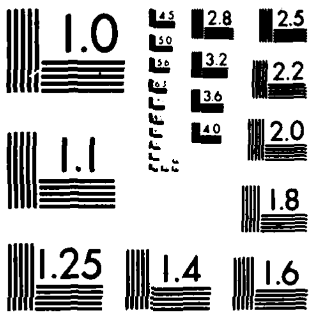

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"ON THE RELIABILITY OF REPAIRABLE SYSTEMS"

by

Mark Brown[1]

The City College, CUNY

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Abstract

The problem of time to first failure for repairable coherent systems of independent exponential components is discussed. Several inequalities are derived and related to previous work of the author and of Keilson to obtain approximations with error bounds for the distribution of the time to first failure.

1. Introduction

The need for quantitative methods to assess system reliability has surfaced in evaluating such diverse systems as nuclear power plants, automatic landing systems and chemical plants. The preliminary steps to system reliability assessment involve identifying the various modes of system failure and assigning parameter valves to the failure and repair distributions. A valuable tool in the identification of system failure is fault tree analysis, a topic of considerable research interest in the engineering literature.

Here, my concern is with the quantitative analysis of reliability for systems with repairable components. This usually follows qualitative analysis, fault tree construction and assignment of parameters. From this input one desires to calculate system characteristics, the most important of which is the distribution of the time to first failure.

In this paper some ideas relevant to the study of time to first failure are discussed. In particular the work of Keilson ([10], [11]) for general coherent systems, is combined with Brown ([4], [5]) and results derived here to obtain approximations with error bounds for the reliability of coherent systems.

Sections 2 - 6 contain background material and mathematical methodology. The main inequalities are presented in Section 7. A numerical example illustrating the precision of the approximation for the parallel case is presented in Section 8.

2. Description of the Mathematical Model

The model which I will discuss appears extensively in the literature (Barlow and Proschan [1], Esary and Proschan [8], Brown [4], Keilson ([10], [11]), Ross [15]). The system has n independent components. Each alternates between independent working periods (often called up periods) and repair (down) periods. The distribution of the up and down periods may vary from component to component. Define:

$$X_i(t) = \begin{cases} 1 & \text{if component i is up at time t} \\ 0 & \text{if component is down at time t} \end{cases}$$

and

$$\underline{X}(t) = (X_1(t),\dots,X_n(t)).$$

Ths system is up at time t if and only if $X(t) \varepsilon G$, a subset of the state space S (S contains the 2^n n-tuples of 0's and 1's). The system is down at time t if and only if $\underline{X}(t) \varepsilon B$, B being the complement of G in S. The set G depends on the structure of the system. For example a k out of n system is one which is up if and only if at least k of its n components are up. The set G in this case consists of all points in S with k or more 1's while B consists of all points in S with k-1 or less 1's.

The k out of n system is an example of a coherent system. A coherent system is defined by the property that $\underline{x} \varepsilon G$ and $\underline{y} \geq \underline{x}$ (meaning $y_i \geq x_i$, i=1,..n) implies $\underline{y} \varepsilon G$. Thus a working system cannot go out of order as a result of repair of one of its down components.

The assumption of coherency is quite reasonable and plays an important role in the analytic treatment of the model.

A further assumption which is often made and which will be followed here is that both up and down periods are exponentially distributed. Under this assumption the process $\{\underline{X}(t), t \geq 0\}$ is a Markov process. As a result the mathematical analysis is simplified, although formidable difficulties remain. The problem of robustness of results derived under exponentiality is of obvious importance.

3. Construction of $\{X(t), t \geq 0\}$

Consider a single component with exponential up times with parameter λ and exponential down times with parameter μ . A convenient construction for $\{X(t), t \geq 0\}$, the zero - one process representing the component state will now be discussed. It was employed in Brown [4] for the study of parallel systems, and is apparently quite well known in other contexts.

Take two independent Poisson processes, process 1 with parameter λ and process 2 with parameter μ . Call the superimposed Poisson process $\{N(t), t \geq 0\}$ and label its event epochs by $T_1, T_2, \ldots$. Define $X(t) = X(0)$ if $N(t) = 0$, otherwise observe $T_{N(t)}$, the last event from the superimposed process at or prior to time t. Set $X(t) = 0$ if $T_{N(t)}$ came from process 1, and $X(t) = 1$ if $T_{N(t)}$ came from process 2. Clearly the constructed process is an alternating renewal process with exponential holding times with parameters λ and μ, and is thus a representation of component behavior.

Next, perform this construction independently for the n components, letting $\underline{X}(0)$ have whatever initial distribution is of interest. We thus have a version of $\{X(t), t \geq 0\}$.

Applications of this construction will follow in Sections 4, 5 and 6.

4. Stochastic Monotonicity

Suppose we want to compare the process $\underline{X}(t)$ under two different initial states α and β, with $\alpha \leq \beta$ $(\alpha_i \leq \beta_i, i = 1,\ldots,n)$. Call the resulting processes $\underline{X}_\alpha(t)$ and $\underline{X}_\beta(t)$. We construct a brivariate version $\{(\underline{X}_\alpha(t), \underline{X}_\beta(t)), t \geq 0\}$ by setting $\underline{X}_\alpha(0) = \alpha$, $\underline{X}_\beta(0) = \beta$ and having the transition mechanism (the 2n independent Poisson processes discussed in Section 3) be identical for both $\underline{X}_\alpha$ and $\underline{X}_\beta$. It immediately follows that $\underline{X}_\alpha(t) \leq \underline{X}_\beta(t)$ for all t under the constructed version. It further follows that for any decreasing set $B(\underline{x} \in B$ and $y \leq \underline{x}$ implies $y \in B)$ that $T_\alpha(B)$, the first passage time to B starting in α, is stochastically smaller than $T_\beta(B)$. Thus for a coherent system (i.e. for B decreasing) the time to first failure is stochastically increasing in the initial state (under the partial ording $\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i$, $i=1,\ldots n)$. A consequence is that $T_{\underline{1}}(B)$ is NBU (where $\underline{1}_i=1$, $i=1,\ldots n)$, a result of Ross [15]. Furthermore $Pr(\underline{X}(t)\epsilon B|\underline{X}(0) = \underline{1})$ is increasing in t.

It is not known whether $T_{\underline{1}}(B)$ is IFRA. Brown and Rao [6] show that the first passage time to a decreasing set for a stochastically monotone Markov chain with monotone paths on a partially ordered finite state space is IFRA, but in the present case the paths are not monotone.

5. Comparison of Perfect State and Steady State

Returning to the construction of $\underline{X}(t)$ given in Section 3, define a component i activity to mean that an event takes place from either of the two Poisson processes associated with component i. Note that a component i activity does not necessarily imply a component i change of state. The waiting time until activity from component i is clearly exponential with parameter $\lambda_i + \mu_i$. Let Z_i denote this random time. Note that $X_i(Z_i + s)$ has the steady state distribution of $X_i(t)$ for all $s \geq 0$, independent of the initial state. Defining $Z = \max_{1 \leq i \leq n} Z_i$ it follows that $\underline{X}(Z + s)$ has the system steady state distribution regardless of the initial system state. Thus system equilibrium, which is not attained for any fixed t, is attained at the random time Z. The random variable Z is distributed as the maximum of n independent exponential random variables with parameters $\lambda_i + \mu_i$, $i = 1,\ldots n$. Denote by $T_E^*(B)$ the first passage time starting from the steady state distribution restricted to G, and let q equal the steady state probability of G. Note that $Pr[T_E > t) = q\ Pr(T_E^* > t)$ for $t \geq 0$.

We suppress the set B in the notation when there is no room for ambiguity. By the stochastic monotonicity, $T_{\underline{1}} \overset{st}{\geq} T_E^*$; by the above construction $T_{\underline{1}} \overset{st}{\leq} Z + T_E$, where Z and T_E are independent. Thus:

$$(1)\ T_E^* \overset{st}{\leq} T_{\underline{1}} \overset{st}{\leq} T_E + Z$$

with Z and T_E independent and $\Pr(T_E > t) = q \Pr(T_E^* > t)$.

Consider the parallel system $(B = \{0\})$. Here, starting in state $\underline{1}$ a visit to B prior to time Z is impossible. Thus:

$$(2)\ T_{\underline{1}}(\underline{0}) \overset{st}{=} Z + T_E(\underline{0})$$

Result (1) is new, while result (2) was obtained by Brown [4]. These inequalities are applied in Section 7.

6. Parallel System

In this section we present an improved and simplified derivation of some of the author's results (Brown [4]) for parallel systems.

Return to the construction of the 0 - 1 process for a single component, with $X(0) = 1$. Recall that as soon as an event occurs from the superimposed Poisson process, the process $X(t)$ enters steady state. Thus $P_{10}(t) = \Pr(X(t) = 0 | X(0) = 1)$, equals the steady state probability of state 0 $(\lambda | \lambda + \mu)$ multiplied by the probability that equilibrium is achieved prior to time $t(1-e^{-(\lambda+\mu)t})$. Thus:

$$(3)\quad P_{10}(t) = \frac{\lambda}{\lambda+\mu}\,(1-e^{-(\lambda+\mu)t})$$

The result (3) is well known. It is usually derived by differential equations.

It immediately follows from (3) that:

$$(4)\quad P_{\underline{1},\underline{0}}(t) = \prod_{i=1}^{n} \frac{\lambda i}{\lambda_i+\mu_i}\,(1 - e^{-(\lambda_i+\mu_i)t})$$

Recall the random variable Z defined in Section 5. Let F_Z denote its cdf. Furthermore, note that $\pi\frac{\lambda i}{\lambda_i+\mu_i}$ is the steady state probability of state $\underline{0}$, which we denote by p. Thus from (4):

$$(5)\quad P_{\underline{1},\underline{0}}(t) = pF_Z(t).$$

Define $\psi_{\underline{1},\underline{0}}$ to be the Laplace transform of $P_{\underline{1},\underline{0}}$, and ψ_Z to be the Laplace transform of Z. It follows from (5) that:

$$(6)\quad \psi_{\underline{1},\underline{0}}(a) = \frac{p\psi_Z(a)}{a}$$

Next, define W to be the waiting time, starting in steady state with the system known to be working, for the first activity from one of the working components (recall that a component activity is an event from the superimposed Poisson process governing that component). Thus, W is conditionally exponentially distributed with parameter $\sum_{\alpha}(\lambda_i+\mu_i)$, given that α is the set of working components. The probability of α given steady state restricted to G is given by $p^*(\alpha) = \frac{1}{q}(\prod_{\alpha}\frac{\lambda_i}{\lambda_i+\mu_i})(\prod_{\tilde{\alpha}}\frac{\lambda_i}{\lambda_i+\mu_i})$, where $q = 1-p$, p defined above. Thus:

$$(7)\quad Pr(W>t) = \frac{1}{q}\sum_{\underline{\alpha}\neq\underline{0}} p^*(\underline{\alpha})\, e^{-t\sum_{\alpha}(\lambda_i+\mu_i)}$$

From (3) and (7) we obtain:

$$(8)\quad P_{\underline{0},\underline{0}}^{(t)} = \prod_{1}^{n}\left[\frac{\lambda_i}{\lambda_i+\mu_i} - \frac{\mu_i}{\lambda_i+\mu_i} e^{-(\lambda_i+\mu_i)t}\right] = qPr(W>t)+p$$

It follows from (8) that

$$(9)\quad \psi_{\underline{0},\underline{0}}^{(a)} = q\left(\frac{1-\psi_W^{(a)}}{a}\right) + \frac{p}{a} = \frac{1-q\psi_W^{(a)}}{a} \quad \text{where}$$

$\psi_{\underline{0},\underline{0}}^{(a)} = 1 + \int_{s=0}^{t} e^{-at} P_{\underline{0},\underline{0}}^{(t)}\, dt$, and ψ_W is the Laplace transform of W.

Furthermore since,

$$(10)\quad P_{\underline{1},\underline{0}}^{(t)} = \int_{s=0}^{t} P_{\underline{0},\underline{0}}^{(t-s)}\, dF_{\underline{1}}(s)$$

it follows that:

$$(11)\quad \psi_{T_{\underline{1}}}^{(a)} = \psi_{\underline{1},\underline{0}}^{(a)} / \psi_{\underline{0},\underline{0}}^{(a)}$$

Combining (6), (8) and (11) we derive the main results:

$$(12)\quad \psi_{T_{\underline{1}}}(a) = \frac{p\psi_Z(a)}{1-q\psi_W(a)}$$

$$(13)\quad \psi_{T_E}(a) = \frac{p}{1-q\psi_W(a)}$$

Note that if N^* is geometric with parameter p, $N=N^*-1$, and $R = \sum_1^N W_i$ where $\{W_i, i \geq 1\}$ is i.i.d. with Laplace transform ψ_W independent of N, then $\psi_R(a) = \frac{p}{1-q\psi_W(a)}$. Thus from (12) and (13), $T_E \overset{st}{=} \sum_1^N W_i$

and $T_{\underline{1}} \overset{st}{=} Z + \sum_1^N W_i$, where Z is independent of $\sum_1^N W_i$.

From (12) and (13), the moments are easily derived (Brown [4]).

I have not been able to find a sample path interpretation for the representation $T_E \overset{st}{=} \sum_1^N W_i$. By following the process at suitable activity epochs I obtain $T_E = \sum_1^M V_i$ where M is a stopping time, Pr(M=0) = p and $V_1 \sim W$. However I find it surprising that this complicated sum behaves like a geometric sum of i.i.d. random variables, and cannot prove it directly (i.e. without Laplace transforms as above).

7. Bounds For Keilson's Approximation

In addition to T_E^* previously defined, Keilson [10] introduced, T_V, the post-recovery exit time. This is obtained by stating the system in steady state restricted to B, and waiting for the first visit to G; the waiting time from this visit to G until the next visit to B is, by definition, the post-recovery exit time. The important facts about T_V are:

(i) ET_V is computable for the general coherent system, while ET_E^* and $ET_{\underline{1}}$ are only known in special cases.

(ii) T_V is completely monotone.

(iii) T_E^* is the stationary renewal distribution corresponding to T_V.

Keilson's insightful approach for the approximation of the reliability of repairable systems is to employ an exponential distribution with mean ET_V as an approximation for T_V, T_E^* and $T_{\underline{1}}$. The quantity ET_V is easily computable, and Keilson motivates the general principle that for highly reliable systems T_V, T_E^* and $T_{\underline{1}}$ should have approximately the same distribution and that this distribution should be approximately exponential.

The approximate exponentiality and equality of distributions needs to be quantified. As T_V and T_E^* are completely monotone (i.e. mixtures of exponential distributions), Keilson looked at the problem of approximating completely monotone distributions by exponential distributions. He proposed that for X completely monotone, that $\rho = \frac{EX^2}{2(EX)^2} - 1$, be used as a distance measure between X and an exponential distribution with the same mean.

Brown [5] , obtained bounds on th distance to exponentiality. Assume that F is IMRL (increasing mean residual life) a larger class than completely monotone, and that G is the stationary renewal distribution corresponding to F. Then:

$$(14)\quad \max(\sup_t|\bar{F}(t) - e^{-t/\mu}|,\ \sup_t|\bar{F}(t) - \bar{G}(t)|,$$
$$\sup_t|\bar{G}(t) - e^{-t/\mu}|,\ \sup_t|\bar{G}(t) - e^{-t/\mu_G}|) \leq \frac{\rho}{\rho+1} = 1 - \frac{\mu}{\mu_G}$$

where $\mu_G = \frac{EX^2}{2EX}$, the mean of G. Moreover the bound for $\sup_t|\bar{F}(t) - e^{-t/\mu}|$ and $\sup|\bar{F}(t) - \bar{G}(t)|$ is sharp even within the subclass of completely monotone distributions.

Returning to first passage times, we let F be the distribution of T_V and G the distribution of T_E^*. Then from (14):

$$(15)\quad \max(\sup_t|\bar{F}_{T_V}(t)-e^{-t/ET_V}|,\ \sup|\bar{F}_{T_E^*}(t)-e^{-t/ET_V}|,$$
$$\sup_t|\bar{F}_{T_V}(t)-\bar{F}_{T_E^*}(t)|,\ \sup_t|\bar{F}_{T_E^*}(t)-e^{-t/ET_E^*}|) \leq 1 - \frac{ET_V}{ET_E^*}$$

To illustrate the parallel case recall that W is the waiting time until activity from a working component, starting in steady state restricted to G (i.e. at least one component is working). The random variable is distributed as a mixture of exponential distributions. It is exponential with parameter $\sum_{\alpha}(\lambda_i+\mu_i)$ with probability $p^*(\alpha)$.

It turns out (Brown [4] p.386) that:

$$(17) \quad ET_V = \frac{q}{p\sum_1^n \mu_i} = \frac{1}{p(EU^{-1})}$$

Moreover since $T_E^* = \sum_1^N W_i$ and $EU = EW$ we have:

$$(18) \quad ET_E^* = \frac{EU}{p} = \frac{EW}{p}$$

It also follows from (14) (since T_E^* is IMRL) that:

$$(16) \quad \sup_t |F_{T_E^*}(t) - e^{-t/ET_E^*}| \leq \frac{\rho_E^*}{\rho_E^*+1}$$

Thus:

$$(19) \quad \rho_V = \frac{ET_E^*}{ET_V} - 1 = (EUEU^{-1}) - 1$$

Furthermore:

$$(20) \quad \rho_E^* = \frac{ET_E^{*2}}{2(ET_E^*)^2} - 1 = p(\frac{EW^2}{2(EW)^2} - 1) = p\rho_W$$

Thus the Keilson distance from exponentiality of T_E^* (ρ_E^*) equals p, the steady state probability of system failure, multiplied by ρ_W, the Keilson distance from exponentiality of W. In both (19) and (20) the ρ parameters are explicitly available in terms of λ_i, μ_i, $i=1, \ldots n$.

Finally, inequalities (1) and (2) relate the approximate exponentiality of T_E and T_V to $T_{\underline{1}}$. From (1):

(21) $\bar{F}_{T_E^*}(t) \le \bar{F}_{T_{\underline{1}}}(t) \le E\bar{F}_{T_E}(t-Z) \le E\bar{F}_{T_E^*}(t-Z)$

From (21) we obtain:

(22) $0 \le \bar{F}_{T_{\underline{1}}}(t) - \bar{F}_{T_E^*}(t) \le E\bar{F}_{T_E^*}(t-Z) - \bar{F}_{T_E^*}(t)$

Since T_E^* is completely monotone its pdf is decreasing. Therefore the interval of length z with highest probability is (0,z]. Thus from (22):

(23) $\bar{F}_{T_{\underline{1}}}(t) - \bar{F}_{T_E^*}(t) \le EF_{T_E^*}(Z)$

But, since T_E^* has a decreasing pdf, F(z) is concave and thus:

(24) $EF_{T_E^*}(Z) \le F_{T_E^*}(EZ)$

Next, since T_E^* is completely monotone it is also DFR (decreasing failure rate). If follows from Brown [5], that:

(25) $F_{T_E^*}(EZ) \le 1-e^{-\left(\frac{EZ}{ET_E^*} + \rho_E^*\right)} \le \frac{EZ}{ET_E^*} + \rho_E^*$

Combining (21) - (25) we arrive at:

(26) $0 \le \bar{F}_{T_{\underline{1}}}(t) - \bar{F}_{T_E^*}(t) \le \frac{EZ}{ET_E^*} + \rho_E^*$ for all t.

Furthermore, it follows from (15) and (26) that:

$$(27)\quad \sup_t |\overline{F}_{T_{\underline{1}}}(t) - e^{-t|ET_V}| \le \frac{\rho_V}{\rho_V+1} + \frac{EZ}{ET_E^*} + \rho_E^*$$

and from (16) and (26) that:

$$(28)\quad \sup_t |\overline{F}_{T_{\underline{1}}}(t) - e^{-t|ET_E^*}| \le \frac{\rho_E^*(\rho_E^*+2)}{(\rho_E^*+1)} + \frac{EZ}{ET_E^*}$$

Lastly, since T_E^* is DFR and has failure rate $1|ET_V$ at $t = 0$, it follows that $\overline{F}_{T_E^*} \ge e^{-t|ET_V}$. Thus from (21) - (24):

$$(29)\quad \sup_t |\overline{F}_{T_{\underline{1}}}(t) - \overline{F}_{T_E^*}(t)| \le 1-e^{-EZ/ET_V} \le \frac{EZ}{ET_V}$$

From (15) and (29) we obtain:

$$(30)\quad \sup_t |\overline{F}_{T_{\underline{1}}}(t) - e^{-t|ET_V}| \le \frac{\rho_V}{\rho_V+1} + \frac{EZ}{ET_V}$$

8. Numerical Example

Consider a parallel system of three identical components with failure rate .01 and repair rate 1.

Here $ET_V = 343{,}433.33$, $ET_E^* = 345{,}181.85$, $\rho_V = .005091$, $\frac{\rho_V}{\rho_V+1} = .005066$, $\rho_E^* = 10^{-8}$, and $EZ = 1.815182$. Consequently $\overline{F}_{T_V}(t)$, $\overline{F}_{T_E^*}(t)$, and $e^{-t|ET_V}$ are all within a distance of .005066 for all t. Furthermore $\sup|\overline{F}_{T_E^*}(t) - e^{-t|ET_E^*}| \le \rho_E^* = 10^{-8}$, and $\sup_t |\overline{F}_{\underline{1}}(t)-e^{-t|ET_E^*}| \le 2\rho_E^* + \frac{EZ}{ET_E^*} = .00000528$.

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